Adaptive Robust Control Design and Experimental Demonstration for Flexible Joint Manipulators

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A control design for flexible joint manipulator in the presence of nonlinearity and mismatched uncertainty is introduced. The control does not need the possible bound of uncertainty a priori. Only the existence of the bound is assumed. A state transformation is introduced via implanted control to tackle a mismatched system. The scheme utilizes the bounding function by combining states and parameters, which is to be estimated. Then an appropriate parameters update laws are designed to guarantee an asymptotic convergency by adopting Lyapunov approach. The control version shows that states converge to zero for the transformed system, and guarantees the uniform stability and boundness. This is also true for the original system in case either the gravitational force is absent or the system is coordinated such that the gravitational force converges to zero as link angles approach zero. The control performance is verified through experiments and shows an enhanced tracking performance for given references.

Key Words : Adaptive Robust Control, Flexible Joint Manipulators, Mismatched System, Uncertainty, Nonlinear Control

1. Introduction

When robots are driven by actuators such as harmonic drivers, the manipulators are usually modeled by connections of rigid links with rigid joints to simplify dynamic motion and subsequent control design. However, the advantage by utilizing joint flexibility in manipulator design is often dedicated to increasing the system performance. For instance, the flexible joint can absorb a certain amount of impact due to accidental collision with environment, which makes the manipulator compliant not to be damaged by external impact, taking a potential advantage over the rigid joints. On the other hand, even if a rigid joint is installed for a transmission using a chain or belt there is inherent flexibility between link and joint, frequently giving rise to undesirable behavior by a simple control design based on rigid joints. However, designing a control for flexible joint manipulators with the inevitable nonlinearity and model uncertainties is extremely difficult.

In this paper, a control scheme for flexible joint manipulators to tackle a nonlinearity and uncertainty is explored. Unfortunately, the uncertainty in the flexible joint manipulators does not satisfy a matching condition which requires that control inputs be appeared in each mode. So far there have been a lot of works related to the study of the control for flexible joint manipulators includ-

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ing singular perturbation (Ficola et al., 1983), and feedback linearization scheme (Khorasani, 1990) which rely on an exact model based approach. Robust adaptive control for uncertain system model is also investigated (Ge, 1996 ; Yim, 2001). The exact model based approach requires exact knowledge of the robot parameters, which is not practically possible. As for robust control based on Lyapunov approach the bound of uncertainty a priori is needed. This may raise practical concern whether we can appropriately estimate the bound of the uncertainty. Insufficient knowledge of the uncertainty may cause unnecessary control cost or saturation in a controller. As an alternative method, adaptive control can be utilized. However, it does not usually guarantee uniform boundedness which is stronger criteria for system performance to overcome the possible large transient response. Basically, the idea of the adaptive control is to reduce the level of uncertainty by estimating unknown parameters. On the other hand, robust control is to design a controller that can tolerate some level of uncertainty and provide satisfactory performance. In many cases, with only adaptive control there may be excessive transient responses even if parameter adaptation converges. Therefore it is worth while to investigate a controller which combines adaptive and robust scheme to enhance system performance. To utilize robust control scheme we have to overcome the mismatched uncertainty issue, which includes flexible joint manipulators. To tackle this mismatched uncertainty, a backstepping based control has been introduced (Yim, 1996 ; Zhi and Khorrami, 2000). In designing the controller the uncertain parameters are required to be time-invariant and the uncertain parameters need to be arranged to form parameterization. Other approaches on the control of flexible joint manipulators are sliding control (Filipescu et al., 2003 ; Huanf and Chen, 2004) and fuzzy and neural network (Amjadi et al., 2001 ; Abdollahi et al., 2003), which mostly utilize those own advantages and show an asymptotic convergence under the limited selection for the bound of uncertain parameter.

The major development of the proposed adaptive robust control in this paper is divided into two parts. A state transformation via implanted control is used for the development. First, by proposing an adaptive version we overcome a practical concern that the possible bound of uncertainty is to be given a priori. Due to insufficient knowledge of the uncertainty, a difficulty in estimating the bound of uncertain parameter may be encountered. Moreover, even if the bound is drawn, it may not be close to the exact bound of the uncertainty, which is likely to make a designer choose a conservative estimation on the bound. As a consequence, unnecessary cost or saturation in the controller can occur. Second, the combined version of adaptive and robust control approach satisfies properties that include uniform stability and uniform boundedness. The adaptive robust control satisfies a property that errors at the transformed states approach zero. Furthermore, by this scheme the original states approach zero in case the gravitational force is absent or the system is coordinated such that gravitational force approaches zero, which is happened at a planar robot. In the following, the procedures to design a control scheme are demonstrated and the control is applied to a 2-link planar flexible joint manipulator, presenting satisfactory experimental results.

2. Flexible Joint Manipulators

Consider an n serial link mechanical manipulator. The links are assumed rigid. The joints are however flexible. All joints are revolute or prismatic and are directly actuated by DC-electric motors. Position of the rigid robot can be described by n generalized coordinates representing the degrees of freedom of the joints. For the flexible joint robot define vectors $q_1=[q^{(2)}\ q^{(4)} \cdots q^{(2n-2)}\ q^{(2n)}]^T$ and $q_2=[q^{(1)}\ q^{(3)} \cdots q^{(2n-3)}\ q^{(2n-1)}]^T$, where $q^{(2)}$, $q^{(4)}\cdots$ are link angles and $q^{(1)}$, $q^{(3)} \cdots$ are joint angles. Let

$$q=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{1}$$

be the 2n-vector of generalized coordinates for the system. The dynamic equation of motion of the flexible joint manipulator can be expressed in terms of the partition of the generalized coordinates (Spong, 1989):

$$\begin{bmatrix} D(q_1) & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\begin{bmatrix} C(q_1,\dot{q}_1)\dot{q}_1 \\ 0 \end{bmatrix}+\begin{bmatrix} G(q_1) \\ 0 \end{bmatrix} + \begin{bmatrix} K(q_1-q_2) \\ -K(q_1-q_2) \end{bmatrix}=\begin{bmatrix} 0 \\ u \end{bmatrix}, \tag{2}$$

where $D(q)$ is the link inertia matrix and J is a constant diagonal matrix representing the inertia of actuator $C(q,\dot{q})\dot{q}$ represents the Coriolis and centrifugal force, $G(q)$ represents the gravitational force, and u denotes the input force from the actuators. K is a constant diagonal matrix representing the torsional stiffness between links and joints (hence K^{-1} exists). We model the joint flexibility by a linear torsional spring at each joint and denote the diagonal matrix of joint stiffness by K. We assume that the rotors are modelled as uniform cylinders so that the gravitational potential energy of the system is independent of the rotor position and is therefore a function only of link position.

3. Adaptive Version of Robust Control

We consider an adaptive version of robust control for a flexible joint manipulator system. This approach is based on the state transformation via implanted control and on combining state vectors and parameters of bounds. It also guarantees uniform stability and uniform boundness, and satisfies a property that transformed states approach zero. This control does not need the bound of the uncertainty a priori. Let $X_1=q_1$, $X_2=\dot{q}_1$, $X_3=q_2$ and $X_4=\dot{q}_2$ also let $x_1=[X_1^T\ X_2^T]^T$, $x_2=[X_3^T\ X_4^T]^T$ and $x=[x_1^T\ x_2^T]^T$. We construct the following two subsystems for the flexible joint manipulator system by using the state variables x_1, x_2:

$$\begin{aligned} N_1: \dot{x}_1(t)&=f_1(x_1(t),\sigma_1(t))+B_1(x_1(t),\sigma_1(t))x_2(t), \\ N_2: \dot{x}_2(t)&=f_2(x(t),\sigma_2(t))+B_2(\sigma_2(t))u(t), \end{aligned} \tag{3}$$

where

$$f_1(x_1,\sigma_1)=\begin{bmatrix} \dot{q}_1 \\ f_{21}(x_1,\sigma_1) \end{bmatrix},$$

$$\begin{aligned} f_{21}&=(x_1,\sigma_1) \\ &=-D^{-1}(q_1,\sigma_1)C(q_1,\dot{q},\sigma_1)\dot{q}_1 \\ &\quad -D^{-1}(q_1,\sigma_1)G(q_1,\sigma_1) \\ &\quad -D^{-1}(q_1,\sigma_1)K(\sigma_1)q_1, \end{aligned}$$

$$f_2(x)=\begin{bmatrix} \dot{q}_2 \\ -J^{-1}(\sigma_2)K(\sigma_2)q_2+J^{-1}(\sigma_2)K(\sigma_2)q_1 \end{bmatrix},$$

$$B_1(x_1,\sigma_1)=\begin{bmatrix} 0 & 0 \\ D^{-1}(\sigma_1,q_1)K(\sigma_1) & 0 \end{bmatrix}, \tag{4}$$

$$B_2(\sigma_2)=\begin{bmatrix} 0 \\ J^{-1}(\sigma_2) \end{bmatrix},$$

Here, $\sigma_1 \in R^{o1}$ and $\sigma_2 \in R^{o2}$ are uncertainty parameter vectors in N_1 and N_2. Suppose we do not need to know the possible bound of uncertainty but the bound should be "compact". Thus, we propose the following Assumption.

Assumption 1. For each subsystem, the mappings $\sigma_1(\cdot): R \rightarrow \Sigma_1 \subset R^{o1}$, $\sigma_1(\cdot): R \rightarrow \Sigma_2 \subset R^{o2}$, $\dot{\sigma}_1(\cdot): R \rightarrow \Sigma_{1t} \subset R^{o1}$ are Lebesgue measurable with $\Sigma_2, \Sigma_2, \Sigma_{1t}$ unknown but compact.

From now on, if no confusion arises we omit argument for the uncertainty in $D(\sigma_1, q_1), C(\sigma_1, q_1, \dot{q}_1)$, etc. Now, we pre-multiply K^{-1} on both sides of the first part of (2) and construct two subsystems as follows:

$$N_1: \hat{D}(q_1)\ddot{q}_1+\hat{C}(q_1,\dot{q})\dot{q}_1+\hat{G}(q_1)+q_1=q_2, \tag{5}$$

$$N_2: J\ddot{q}_2+K(q_2-q_1)=u, \tag{6}$$

where

$$\begin{aligned} \hat{D}(q_1)&=K^{-1}D(q_1), \\ \hat{C}(q_1,\dot{q}_1)&=K^{-1}C(q_1,\dot{q}_1), \\ \hat{G}(q_1)&=K^{-1}G(q_1). \end{aligned} \tag{7}$$

The problem is to design control u which makes the systems N_1, N_2 have good performance. Notice that the uncertainty does not meet the matching condition (Chen and Leitmann, 1987) of the total system. Thus, we divide the total system into two subsystems as shown in (5) and (6) and introduce a virtual (or implanted) control for the subsystem N_1. Therefore both subsytems have "inputs". Let us rewrite (5) and (6) as

$$N_1: \hat{D}(q_1)\ddot{q}_1+\hat{C}(q_1,\dot{q}_1)\dot{q}_1+\hat{G}(q_1)+q_1 \\ =u_1+q_2-u_1, \tag{8}$$

$$N_2: J\ddot{q}_2+K(q_2-q_1)=u, \tag{9}$$

where the "control" u_1 is implanted. This does not affect the dynamics in N_1. We now transform the system (N_1, N_2) to a system $(\hat{N}_1, \hat{N}_2)$ by using a state transformation. First, let $z_1=[Z_2^T\ Z_2^T]^T$, $z_2=[Z_3^T\ Z_4^T]^T$ and $z=[z_1^T\ z_2^T]^T$ where

$$\begin{aligned} Z_1&=q_1,\\ Z_2&=\dot{q}_1,\\ Z_3&=q_2-u_1,\\ Z_4&=\dot{q}_2-\dot{u}_1, \end{aligned} \tag{10}$$

This implies that $z_1=x_1$ and $z_2=x_2-[u_1\ \dot{u}_1]^T$. The dynamics of the manipulator can be expressed in terms of z:

$$\hat{N}_1: \hat{D}(Z_1)\ddot{Z}_1=-\hat{C}(Z_1,\dot{Z}_1)\dot{Z}_1-\hat{G}(Z_1) \\ -Z_1+Z_3+u_1, \tag{11}$$

$$\hat{N}_2: J\ddot{Z}_3=-J\ddot{u}_1-KZ_3+KZ_1-Ku_1+u. \tag{12}$$

Let

$$\begin{aligned} &\phi_1(q_1,\dot{q}_1,\sigma_1,\dot{\sigma}_1)\\ &=-\frac{1}{2}\dot{\hat{D}}(q_1,\dot{q}_1,\sigma_1,\dot{\sigma}_1)(\dot{q}_1+S_1q_1)\\ &\quad-\hat{C}(q_1,\dot{q}_1,\sigma_1)\dot{q}_1\\ &\quad-\hat{G}(q_1,\sigma_1)-q_1+\hat{D}(q_1,\sigma_1)S_1\dot{q}_1, \end{aligned} \tag{13}$$

for given $S_1=diaj[S_{li}]_{n\times n}$, $S_{li}>0$. Then, we see that there exists an uncertain function $\rho_1: R^n\times R^n \rightarrow R_+$ such that for all $q_1\in R^n$, $\dot{q}_1\in R^n$, $\sigma_1\in \sum_1$, $\dot{\sigma}_1\in \sum_{1t}$,

$$\|\phi_1(q_1,\dot{q}_1,\sigma_1,\dot{\sigma}_1)\|\le\rho_1(q_1,\dot{q}_1). \tag{14}$$

Assumption 2. (Corless and Leimann, 1984)

(1) There exist an unknown constant vector $\beta_1\in(0,\infty)$ and a known function $\prod_1: R^n\times R^n\times (0,\infty)^k \rightarrow R_+$ such that for all $q_1\in R^n, \dot{q}_1\in R^n$,

$$\rho_1(q_1,\dot{q}_1)=\Pi_1(q_1,\dot{q}_1,\beta_1). \tag{15}$$

(2) The function $\prod_1(q_1,\dot{q}_1,\cdot): (0,\infty)^k \rightarrow R_+$ is C^2 (i.e., 2-times continuously differentiable) and concave (i.e., $-\prod_1(q_1,\dot{q}_1,\cdot)$ is convex), and non-decreasing with respect to each coordinate of argument, β_1.

(3) The functions $\prod_1(\cdot)$ and $\frac{\partial\Pi_1}{\partial\beta_1}(\cdot)$ are both continuous.

Let

$$\begin{aligned} \mu_1&=(\dot{Z}_1+S_1Z_1)^T\Pi_1(z_1,\hat{\beta}_1),\\ \mu_1&=[\mu_{11}\ \mu_{12}\ \cdots\ \mu_{1n}]^T,\\ p_1&=[p_{11}\ p_{12}\ \cdots\ p_{1n}]^T. \end{aligned} \tag{16}$$

We construct controller for the subsystem

$$u_1(t)=-K_{p1}Z_1(t)-K_{v1}\dot{Z}_1(t) \\ +p_1(z_1(t),\hat{\beta}_1(t),\varepsilon_1(t)), \tag{17}$$

Where

$$\begin{aligned} K_{p1}&=diag\lfloor K_{pli}\rfloor_{n\times n},\ k_{pli}>0,\\ K_{v1}&=diag[K_{vli}]_{n\times n},\ k_{vli}>0, \end{aligned} \tag{18}$$

p_{1i} is chosen to be

$$p_{1i}=\begin{cases} -\dfrac{\mu_{1i}}{\|\mu_{1i}\|}\Pi_1(z_1,\hat{\beta}_1) & \text{if } \|\mu_{1i}\|\le\varepsilon_1\\ -\sin\left(\dfrac{\pi\mu_{1i}}{2\varepsilon_1}\right)\Pi_1(z_1,\hat{\beta}_1) & \text{if } \|\mu_{1i}\|\le\varepsilon_1, \end{cases} \tag{19}$$

$i=1,2,\cdots,n$. Note that

$$p_{li}=\begin{cases} \le-\dfrac{\mu_{1i}}{\varepsilon_1}\Pi_1(z_1,\hat{\beta}_1) & \text{if } 0\le\mu_{1i}\le\varepsilon_1\\ \ge-\dfrac{\mu_{1i}}{\varepsilon_1}\Pi_1(z_1,\hat{\beta}_1) & \text{if } -\varepsilon_1\le\mu_{1i}<0, \end{cases} \tag{20}$$

and $\|p_{li}\|\le\Pi_1(y_1,\hat{\beta}_1)$. The parameter update law for $\hat{\beta}_1$ is determined as

$$\begin{aligned} \dot{\hat{\beta}}_1(t)&=T_1^{-1}\|\dot{Z}_1+S_1Z_1\|\frac{\partial\Pi_1^T}{\partial\beta_1}(z_1,\hat{\beta}_1)\\ &\hat{\beta}_1(t_0)\in(0,\infty)^k, \end{aligned} \tag{21}$$

where T_1 is a nonsingular diagonal matrix with positive elements, and n corresponds to the number of links. The control parameter $\varepsilon_1(\cdot)$ is set by

$$\dot{\varepsilon}_1(t)=-\frac{n}{4l_1}\varepsilon_1(t),\ \varepsilon_1(t_0)\in(0,\infty),\ l_1>0, \tag{22}$$

Here, $\varepsilon_1(t)$ along with $\varepsilon_1(t_0)$ needs to be determined not to be close to zero $t\rightarrow 0$ by adjusting l_1 in real implementation, which prevents a possible chattering and undefined value on control p_{li}.

Moreover, we know that $\hat{\beta}_1(t)>0$, for all $t\ge t_0$ if $\hat{\beta}_1(t_0)>0$ which satisfies Assumption 2(1).

The selection of K_{p1}, K_{v1} can be conducted as follows.

1) After choosing s_1, select $\underline{\lambda}_1$ such that for $w_1 > 0$,

$$\underline{\lambda}_1 - \frac{1}{2}\lambda_{\max}(\bar{S}_1) > 0, \tag{23}$$

Where

$$\underline{\lambda}_1 = \min\{\lambda_{\min}(K_{v1}),\ \lambda_{\min}(S_1 K_{p1})\}, \tag{24}$$

$$\bar{S}_1 = \begin{bmatrix} S_1^2 & S_1 \\ S_1 & I \end{bmatrix}. \tag{25}$$

2) Based on $\underline{\lambda}_1$ we select the values for K_{v1}, K_{p1}. Next, let

$$\begin{aligned} &\phi_2(z_1, z_2, \sigma_1, \sigma_2, \dot{\sigma}_1) \\ &:= -J(\sigma_2)\ddot{u}_1(z_1, z_2, \sigma_1, \sigma_2) \\ &\quad - K(\sigma_2) Z_3 + K(\sigma_2) Z_1 \\ &\quad - K(\sigma_2) u_1 + J(\sigma_2) S_2 \dot{Z}_3, \end{aligned} \tag{26}$$

for given $S_2 = diag[S_{2i}]_{n\times n}$, $S_{2i} > 0$. Then, we see that there exists an uncertain function $\sigma_2 : R^{2n} \times R^{2n} \to R_+$, such that for all $z_1 \in R^{2n}$, $z_2 \in R^{2n}$, $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$, $\dot{\sigma}_1 \in \Sigma_{1t}$,

$$\|\phi_2(z_1, z_2, \sigma_1, \sigma_2, \dot{\sigma}_1)\| \le \rho_2(z_1, z_2). \tag{27}$$

Assumption 3. (Corless and Leitmann, 1984)

(4) There exist an unknown constant $\beta_2 \in (0, \infty)^j$ and a known function $\Pi_2 : R^{2n} \times R^{2n} \times (0, \infty)^j \to R_+$ such that for all $z_1 \in R^{2n}$, $z_2 \in R^{2n}$,

$$\rho_2(z_1, z_2) = \Pi_2(z_1, z_2, \beta_2). \tag{28}$$

(5) The function $\Pi_2(z_1, z_2, \cdot) : (0, \infty)^j \to R_+$ is C^1, concave and non-decreasing with respect to each coordinate of argument, β_2.

(6) The functions $\Pi_2(\cdot)$ and $\frac{\partial \Pi_2}{\partial \beta_2}(\cdot)$ are both continuous. Next, for given $\varepsilon_2 > 0$ we design control for the subsystem $\hat{N}_2$ as follows:

$$\begin{aligned} u(t) = &- K_{p2} Z_3(t) - K_{v2} \dot{Z}_3(t) \\ &+ p_2(z_1(t), z_2(t), \hat{\beta}_2(t), \varepsilon_2(t)), \end{aligned} \tag{29}$$

Where

$$p_2(z_1, z_2, \hat{\beta}_2, \varepsilon_2) = \begin{cases} -\dfrac{\mu_2(z_1, z_2, \hat{\beta}_2)}{\|\mu_2(z_1, z_2, \hat{\beta}_2)\|}\Pi_2(z_1, z_2, \hat{\beta}_2) & \text{if } \|\mu_2(z_1, z_2, \hat{\beta}_2)\| > \varepsilon_2 \\ -\dfrac{\mu_2(z_1, z_2, \hat{\beta}_2)}{\varepsilon_2}\Pi_2(z_1, z_2, \hat{\beta}_2) & \text{if } \|\mu_2(z_1, z_2, \hat{\beta}_2)\| > \varepsilon_2, \end{cases} \tag{30}$$

$$\mu_2(z_1, z_2, \hat{\beta}_2) = (\dot{Z}_3 + S_2 Z_3)\Pi_2(z_1, z_2, \hat{\beta}_2). \tag{31}$$

The update law of parameter $\hat{\beta}_2$ and control parameter ε_2 are designed as follows.

$$\dot{\hat{\beta}}_2(t) = T_2^{-1}\frac{\partial \Pi_2^T}{\partial \beta_2}(z_1, z_2, \hat{\beta}_2)\|\dot{Z}_3(t) + S_2 Z_3(t)\|, \tag{32}$$

$$\begin{aligned} &\dot{\varepsilon}_2 = -\frac{1}{4 l_2}\varepsilon_2(t) \\ &\hat{\beta}_2(t_0) \in (0, \infty)^j,\ \varepsilon_2(t_0) \in (0, \infty),\ l_2 > 0, \end{aligned} \tag{33}$$

where T_2 is a positive diagonal matrix. Again, $\varepsilon_2(t)$ along with $\varepsilon_2(t_0)$ needs to be determined not to be close to zero $t \to 0$ by adjusting l_2, which is the same argument on $\varepsilon_1(t)$. Here, we know that $\hat{\beta}_2(t) > 0$, for $t \ge t_0$, if $\hat{\beta}_2(t_0) > 0$, which satisfies Assumption 3(4).

The selection of K_{p2} and K_{p2} is shown as the following subsequent steps.

1) Let

$$\underline{\lambda}_2 = \min\{\lambda_{\min}(K_{v2}),\ \lambda_{\min}(S_2 K_{p2})\} \tag{34}$$

2) After choosing S_2, select $\underline{\lambda}_2$ such that for $w_1 > 0$,

$$\underline{\lambda}_2 - \frac{1}{2} w_1^{-1} > 0. \tag{35}$$

3) Based on $\underline{\lambda}_2$ we select K_{v2}, K_{p2}.

Remark 1 The control $u(t)$ relies on the acceleration and signal. This can be undesirable due to possible noise or signal contamination. However, when the control action is implemented in experiment, the acceleration can be adapted by computing the simple difference equation in a digital way as follows, which is excluding the complex computation from the dynamic model. Then, the other remain terms can be put into the uncertainty terms, and eventually the adaptive robust control $u(t)$ counts for uncertainty by updating the uncertain parameters at bounding function. The experimental results shown later illustrate the justification of this method. Other alternative for avoiding a possible concern on acceleration feedback is to install a low pass filter after the computed control $u(t)$, which is dedicated to attenuating high frequency components caused by differentiating velocities of joint and link angles.

$$\dot{Z}_3=\dot{q}_2-\dot{u}_1 \\ =\frac{q_2(t-\Delta t)-q_2(t)}{\Delta t}-\frac{u_1(t+\Delta t)-u_1(t)}{\Delta t} \quad (36)$$

To avoid the differentiations of signals, a low pass filter design can be one of solutions. Viewing the controller proposed in this article (29) there exists many differentiation terms which can produce the chattering or noise amplification. In stead of directly employing the numerical difference, we introduce a filter dynamics for the differentiation terms as following manner.

$$\frac{1}{\beta}\frac{d\dot{q}_{2f}}{dt}+\dot{q}_{2f}=\dot{q}_2 \\ \frac{1}{\beta}\frac{d\dot{u}_{1f}}{dt}+\dot{u}_{1f}=\dot{u}_1 \quad (37)$$

where $\dot{q}_{2f}$ and $\dot{u}_{1f}$ represent the filtered derivative value of joint angle and virtual input, respectively after passing through the filter. The input signals to the filter are obtained from the numerical difference of the encoder signals, and virtual input. β is the time constant. On the other hand, keeping the derivative signal in control $u(t)$ without using the filter can be also suitably implemented because the measured signal can be embedded into the bounding function of ϕ_2:

$$u(t)=-K_{p2}Z_3-K_{v2}\dot{Z}_3+p_2 \\ =-K_{p2}Z_3-K_{v2}(\dot{q}-\dot{u}_1)+p_2 \\ =-K_{p2}Z_3-K_{v2}(\dot{q}_2-\dot{u}_1)_n \\ +K_{v2}(\dot{q}_2-\dot{u}_1)_{un}+p_2 \quad (38)$$

where $(\dot{q}_2-\dot{u}_1)_n$ and $(\dot{q}_2-\dot{u}_1)_{un}$ represent the nominal value calculated from the direct difference equation and noise-involved part, respectively. The $(\dot{q}_2-\dot{u}_1)_{un}$ part can be put into the modified function ϕ_2, yielding to a modified bounding function $\rho_2(\cdot)$. However, there might be a question how we can estimate the bound of $(\dot{q}_2-\dot{u}_1)_{un}$. In a theoretical basis, this can be set by assigning a reasonable bounding function (affine or polynomial).

Assumption 4. There exist unknown positive constants $\underline{\sigma}_K$, $\bar{\sigma}_K$ such that

$$\underline{\sigma}_k I\le\hat{D}(\sigma_1,q_1)\le\bar{\sigma}_k I,\ \forall q_1\in R^n,\ \forall\sigma_1\in\textstyle\sum_1. \quad (39)$$

Define the parameter estimate vectors

$$\hat{\psi}_1(t)=[\hat{\beta}_{11}(t)\ \hat{\beta}_{21}(t)\ \cdots\ \hat{\beta}_{k1}(t)\ \varepsilon_1(t)]^T \\ \in(0,\infty)^{k+1}=:\psi_1, \\ \hat{\psi}_2(t)=[\hat{\beta}_{12}(t)\ \hat{\beta}_{22}(t)\ \cdots\ \hat{\beta}_{j2}(t)\ \varepsilon_2(t)]^T \\ \in(0,\infty)^{j+1}=:\psi_2, \\ \hat{\psi}=[\hat{\psi}_1^T\ \hat{\psi}_2^T]^T, \\ \psi=\psi_1\cup\psi_2, \quad (40)$$

and the parameter vectors

$$\psi_1=[\beta_{11}\ \beta_{21}\ \cdots\ \beta_{k1}\ 0]^T>0, \\ \psi_2=[\beta_{12}\ \beta_{22}\ \cdots\ \beta_{j2}\ 0]^T>0. \quad (41)$$

The controlled system can be described by

$$\hat{D}(Z_1)\ddot{Z}_1=-\hat{C}(Z_1,\dot{Z}_1)\dot{Z}_1-\hat{G}(Z_1)-Z_1+Z_3+u_1, \\ J\ddot{Z}_3=-J\ddot{u}_1-KZ_3-KZ_1-Ku_1+u, \\ \dot{\hat{\psi}}(t)=\begin{bmatrix} T_1^{-1}\|Z_1(t)+S_1Z_1(t)\|\frac{\partial\Pi_1^T}{\partial\beta_1}(z_1(t),\hat{\beta}_1(t)) \\ -\frac{n}{4l_1}\varepsilon_1(t) \\ T_2^{-1}\|\dot{Z}_3(t)+S_2Z_3(t)\|\frac{\partial\Pi_2^T}{\partial\beta_2}(z_1(t),z_2(t),\hat{\beta}_2(t)) \\ -\frac{1}{4l_2}\varepsilon_2(t)\end{bmatrix}. \quad (42)$$

Here, arguments on the uncertainty in $\hat{D}$, $\hat{C}$, $\hat{G}$, and J are omitted for simplicity.

Theorem 1. Suppose Assumptions 1-3 are met, then the system (42) under the control (29) has the following properties.

Property 1. Existence of Solutions. For each $(z_0,\hat{\psi}_0,t_0)\in R^{4n}\times\psi\times R$ there exists a solution $(z,\hat{\psi}):[t_0,t_1)\to R^{4n}\times\psi$ of (42) with $(z(t_0),\hat{\psi}(t_0))=(z_0,\psi_0)$.

Property 2. Uniform Stability. For each $\eta>0$ there exists $\delta>0$ such that if $(z(\cdot),\hat{\psi}(\cdot))$ is any solution of (42) with $\rho_1(z_1)$ then $\|z(t)\|$, $\|\hat{\psi}(t)-\psi\|<\eta$ for all $t\in[t_0,t_1)$.

Property 3. Uniform Boundedness of Solutions. For each $r_1,r_2>0$ there exist $d_1(r_1,r_2)$, $d_2(r_1,r_2)\ge0$ such that if $(z(\cdot),\hat{\psi}(\cdot))$ is any solution of (42) with $\|z(t_0)\|\le r_1$ and $\|\hat{\psi}(t_0)-\psi\|\le r_2$ then $\|z(t)\|\le d_1(r_1,r_2)$ and $\|\psi(t)-\psi\|\le d_2(r_1,r_2)$ for all $t\in[t_0,t_1)$.

Property 4. Extension of Solutions: Every solution of (42) can be extended into a solution defined on $[t_0, \infty]$.

Property 5. Convergence of $z(\cdot)$ to 0. If (z, ψ): $[t_0, \infty] \rightarrow R^{4n} \times \Psi$ is a solution of (42) then

$$\lim_{t\rightarrow\infty} z(t) = 0 \tag{43}$$

Proof. The detail proof is shown at the Appendix.

Theorem 2. Suppose that Assumptions 1-4 are met, then the system (5-6), (21-22), and (32-33) under the control (29) satisfies Properties 1-4.

Proof. The system performance can be shown similarly to (Kim and Oho, 2006) by investigating the original system (N_1, N_2).

Lemma 1. If either the gravitational force $G(q_1)$ is absent (i.e., $G(q_1) \equiv 0$) or system (5-6) is coordinated such that gravitational force approaches zero as q_1 converges to zero (i.e., $G(q_1) \rightarrow 0$ as $q_1 \rightarrow 0$), then the system (5-6), (21-22), and (32-33) under the control (29) satisfies Properties 1-5.

Proof. We have shown that Properties 1-4 are satisfied in Theorem 2. We need to investigate Property 5. We know that $z(t) \rightarrow 0$ as $t \rightarrow \infty$, this means that $x_1 = z_1 \rightarrow 0$, and $z_2 \rightarrow 0$. For the state X_3, we have

$$\begin{aligned} X_3 &= Z_3 + u_1, \\ X_4 &= \dot{Z}_3 + \dot{u}_2. \end{aligned} \tag{44}$$

Here, we decompose $\rho_1(z_1)$ as two bounded functions, which one is related to gravitational force and the other is related to other forces:

$$\rho_1(z_1) = \Pi_1(z_1, \hat{\beta}_1) = (\kappa_0^T(z_1) + k^T(z_1))\beta_1 \tag{45}$$

where $\kappa_0(\cdot)$ and $\kappa(\cdot)$ are vector functions which belong to a class K function and $\kappa_0^T(z_1)\beta_1$ is corresponding to gravitational force $\|\hat{G}(q_1)\|$ If $z_1 \rightarrow 0$, then $\kappa_0 \rightarrow 0$ by the condition imposed in Lemma 1, and the second term in (45) converges to zero. This yields $\Pi_1(z_1, \hat{\beta}_1) \rightarrow 0$ Hence $\rho_1 \rightarrow 0$, as $t \rightarrow \infty$ Therefore u_1 approaches zero as $t \rightarrow \infty$ and thus $X_3 \rightarrow 0$. For the state X_4, by computing $\dot{u}_1$ we have the following result.

$$\begin{aligned} \dot{u}_1 = &-\left(K_{p1} - \frac{\partial p_1}{\partial Z_1}\right)\dot{Z}_1 - \left(K_{v1} - \frac{\partial p_1}{\partial \dot{Z}_1}\right)\ddot{Z}_1 \\ &+ \frac{\partial p_1}{\partial \hat{\beta}_1}\dot{\hat{\beta}}_1 + \frac{\partial p_1}{\partial \varepsilon_1}\dot{\varepsilon}_1. \end{aligned} \tag{46}$$

Since $Z_1, \dot{Z}_1 \rightarrow 0$ as $t \rightarrow \infty$, we see that the first and the third term converge to zero. For the second term in (11)

$$\ddot{Z}_1 = -\hat{D}(Z_1)^{-1}(\hat{C}(Z_1, \dot{Z}_1)\dot{Z}_1 - \hat{G}(Z_1) - Z_1 + Z_3 + u_1). \tag{47}$$

We see that $\ddot{Z}_1 \rightarrow 0$ since $z \rightarrow 0$. The last term in (46) converges to zero since $\varepsilon_1(\cdot)$ is bounded and $\dot{\varepsilon}_1(\cdot)$ is uniformly continuous, $\dot{\varepsilon}_1 \rightarrow 0$ by Barbalat's Lemma. Therefore $X_4 \rightarrow 0$. This proves that $\lim_{t\rightarrow\infty} x(t) \rightarrow 0$ as $t \rightarrow \infty$.

4. Experimental Verification

Consider a 2-link flexible joint manipulator (Fig. 1). We use the two adaptive versions of robust control. Let link angle vectors $q_1 = [q^{(2)}\ q^{(4)}]^T$ and joint angle vectors $q_2 = [q^{(1)}\ q^{(3)}]^T$. Then we have $D(q_1)$, $C(q_1, \dot{q}_1)$, $G(q_1)$, J, K as follows and all parameters are unknown.

$$\begin{aligned} D(q_1) &= \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \\ C(q_1, \dot{q}_1) &= \begin{bmatrix} -m_2 l_1 l_2 \sin q^{(4)} \dot{q}^{(4)} & -m_2 l_1 l_2 \sin q^{(4)}(\dot{q}^{(4)} + q^{(2)}) \\ m_2 l_1 l_2 \sin q^{(4)} \dot{q}^{(2)} & 0 \end{bmatrix}, \\ G(q_1) &= \begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \sin q^{(2)} + m_2 l_{c2} \sin(q^{(2)} + q^{(4)}) \\ m_2 l_{c2} \sin(q^{(2)} + q^{(4)}) \end{bmatrix}, \\ J &= \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \end{aligned} \tag{48}$$

where

$$\begin{aligned} d_{11} &:= m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{2c} \cos q^{(4)}) \\ &\quad + m_2 l_{c1}^2 + I_1 + I_2, \\ d_{11} &:= m_2(l_{c2}^2 + l_1 l_{2c} \cos q^{(4)}) + I_2, \\ d_{11} &:= d_{12}, d_{22} := m_2 l_{c2}^2 + I_2. \end{aligned} \tag{49}$$

Select each values for the two cases as follows:

$$S_1 = S_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ q_k = 1, \ T_1 = T_2 = I_{2\times2}, \tag{50}$$

We choose $\varepsilon_1 = \varepsilon_2 = 10$ for the case 1. Based onthe above values we can choose $\underline{\lambda}_1 = 1$, and $\underline{\lambda}_2 = 2$, to satisfy (24) and (35). So we select $K_{p1}=1$, $K_{v1}=2$, $K_{p2}=2$ and $K_{v2}=3$. Let $q_d^{(2)}$, $q_d^{(4)}$ be the desired positions of links and $q_d^{(1)}$, $q_d^{(3)}$ be the desired joint angles. We want links to be tracked to the desired trajectory. Let $\tilde{q}^{(2)}=q^{(2)}-q_d^{(2)}$ and $\tilde{q}^{(4)}=q^{(4)}-q_d^{(4)}$. Also the joint angles errors are written as $\tilde{q}^{(1)}=q^{(1)}-q_d^{(1)}$ and $\tilde{q}^{(3)}=q^{(3)}-q^{(3)}$.

Next, set

$$\Pi_1(z_1,\hat{\beta}_1)=\hat{\beta}_{11}(\tilde{q}^{(2)2}+\tilde{q}^{(4)2})+\hat{\beta}_{21}(\dot{\tilde{q}}^{(2)2}+\dot{\tilde{q}}^{(4)2}), \quad (51)$$

$$\begin{aligned}\Pi_2(z_1,z_2,\hat{\beta}_e)=&\hat{\beta}_{12}(\tilde{q}^{(2)2}+\tilde{q}^{(4)2})+\hat{\beta}_{22}(\tilde{q}^{(1)2}+\tilde{q}^{(3)2})\\&+\hat{\beta}_{32}(\dot{\tilde{q}}^{(2)2}+\dot{\tilde{q}}^{(4)2})+\hat{\beta}_{42}(\dot{\tilde{q}}^{(1)2}+\dot{\tilde{q}}^{(3)2})\end{aligned} \quad (52)$$

Now we have following controllers:

$$u_1=-K_{p1}\tilde{q}_1-K_{v1}\dot{\tilde{q}}+p_1, \quad (53)$$

$$p_1=[p_{11}\ p_{12}]^T \quad (54)$$

$$p_{11}=\begin{cases}-\dfrac{u_{11}}{\|u_{11}\|}\Pi_1(\tilde{q},\hat{\beta}_1), & \text{if } \|u_{11}\|>\varepsilon_1,\\ -\sin\left(\dfrac{\pi u_{11}}{2\varepsilon_1}\right)\Pi_1(\tilde{q}_1,\hat{\beta}_1), & \text{if } \|u_{11}\|\le\varepsilon_1\end{cases} \quad (55)$$

$$p_{12}=\begin{cases}-\dfrac{u_{12}}{\|u_{12}\|}\Pi_1(\tilde{q},\hat{\beta}_1), & \text{if } \|u_{12}\|>\varepsilon_1,\\ -\sin\left(\dfrac{\pi u_{12}}{2\varepsilon_1}\right)\Pi_1(\tilde{q}_1,\hat{\beta}_1), & \text{if } \|u_{12}\|\le\varepsilon_1\end{cases} \quad (56)$$

$$[u_{11}\ u_{12}]^T=(\dot{\tilde{q}}+S_1\dot{\tilde{q}}_1)^T\Pi_1(\tilde{q},\hat{\beta}_1), \quad (57)$$

$$u=-K_{p2}(\tilde{q}_2-u_1)-K_{v2}(\dot{\tilde{q}}_2-\dot{u}_1)+p_2, \quad (58)$$

where

$$p_2=\begin{cases}-\dfrac{\mu_2}{\|\mu_2\|}\Pi_2, & \text{if } \|\mu_2\|>\varepsilon_2,\\ -\dfrac{\mu_2}{\varepsilon_2}\Pi_2, & \text{if } \|\mu_2\|>\varepsilon_2\end{cases} \quad (59)$$

$$u_2=(\dot{\tilde{q}}_2-\dot{u}_1+S_2(\tilde{q}_2-u_1))\Pi_2(z_1,z_2,\hat{\beta}_2). \quad (60)$$

The update laws of parameters for the control scheme are shown:

$$\begin{aligned}\dot{\hat{\beta}}_1&=[\dot{\hat{\beta}}_{11}\ \dot{\hat{\beta}}_{21}]^T\\&=\begin{bmatrix}\|\dot{\tilde{Z}}_1+S_1\tilde{Z}_1\|(q^{(2)2}+q^{(4)2})\\ \|\dot{\tilde{Z}}_1+S_1\tilde{Z}_1\|(\dot{\tilde{q}}^{(2)2}+\dot{\tilde{q}}^{(4)2})\end{bmatrix}\end{aligned} \quad (61)$$

$$\begin{aligned}&=[\dot{\hat{\beta}}_{12}\ \dot{\hat{\beta}}_{22}\ \dot{\hat{\beta}}_{32}\ \dot{\hat{\beta}}_{42}]^T\\&=\begin{bmatrix}\|\dot{\tilde{Z}}_3+S_2\tilde{Z}_3\|(\tilde{q}^{(2)2}+\tilde{q}^{(4)2})\\ \|\dot{\tilde{Z}}_3+S_2\tilde{Z}_3\|(\tilde{q}^{(1)2}+\tilde{q}^{(4)2})\\ \|\dot{\tilde{Z}}_3+S_2\tilde{Z}_3\|(\dot{\tilde{q}}^{(2)2}+\dot{\tilde{q}}^{(4)2})\\ \|\dot{\tilde{Z}}_3+S_2\tilde{Z}_3\|(\dot{\tilde{q}}^{(1)2}+\dot{\tilde{q}}^{(3)2})\end{bmatrix}\end{aligned} \quad (62)$$

$\varepsilon_2(\cdot)$, $\varepsilon_2(\cdot)$ are chosen as:

$$\dot{\varepsilon}_1=-\frac{2}{1.5}\varepsilon_1, \quad (63)$$

$$\dot{\varepsilon}_2=-\frac{2}{1.5}\varepsilon_2. \quad (64)$$

The designed 2-link manipulator consists of links, tortional springs with different stiffness installed between motors and links, electric motors, and motion controller. The experimental setup for 2-link manipulator with flexibility is illustrated in Fig. 2. Basically, the flexibility between motor shaft and link exists, which implies that the link and motor angles are independently measured to employ the proposed the adaptive robust control algorithm. We placed four encoders to measure the link angles and joint angles for the two degree of freedom manipulator. The joint angles of two motors are directly measured by encoders installed on the axes of the motor shafts. On the other

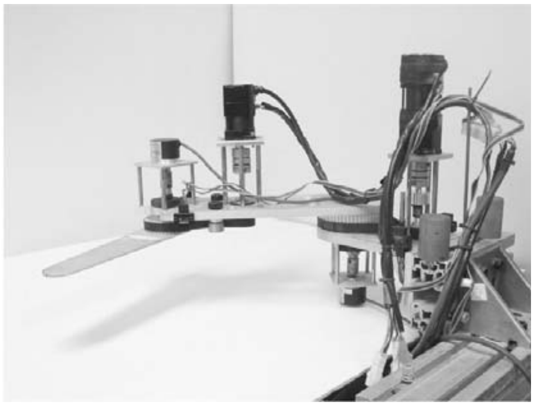

Fig. 1 Experimental setup for a flexible joint manipulator

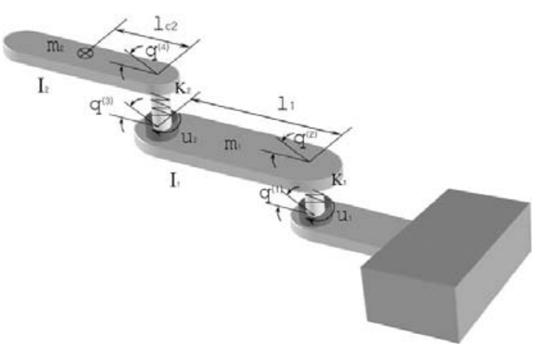

Fig. 2 Model of 2-link flexible joint manipulator

hand, the link angles are measure by installing pulleys between the rotating shafts and links, which are rotated by timing belts. The devices are well illustrated in Fig. 1. For the experimental setup, all parameters are roughly given, which are not necessarily known : m_1=2.8 kg, m_2=0.3 kg, l_1=0.25 m, l_{c2}=0.1 m, and K_1=1.48 kgf·m, K_2= 1.15 kgf·m, I_1=0.112 kgm^2, I_2=0.003 kgm^2, J_1= J_2=0.001 kgm^2, g=9.8 m/s^2 and the control parameters ate assigned in the followings : $K_{p11}=K_{p21}=20$, $K_{p12}=K_{p22}=10$, $K_{v11}=K_{v21}=10$, $K_{v12}=K_{v22}=5$, $s_{11}=s_{12}=1$, $s_{21}=s_{22}=1$, $\varepsilon_1=5$, $\varepsilon_2=5$, $\beta_1=5$ is chosen according to the selection of β_1 based on above parameters.

Figures 3 and 4 show the step and sinusoidal responses, respectively by simply applying PID control on the assumption that the joints are rigid. Viewing the results, as we expect, the control performance is almost perfect. This implies that when the joint flexibility is rigid enough, the manipulator is easily controlled. On the other hand, in case of flexibility in joints, which is shown in Figs. 5∼8 the experimental results for the control performance only by PID control are not satisfactory with regards to large steady error and phase lag. The experimental comparisons between PID and adaptive robust control are illustrated from Figs. 5∼8 for step and sinusoidal in-

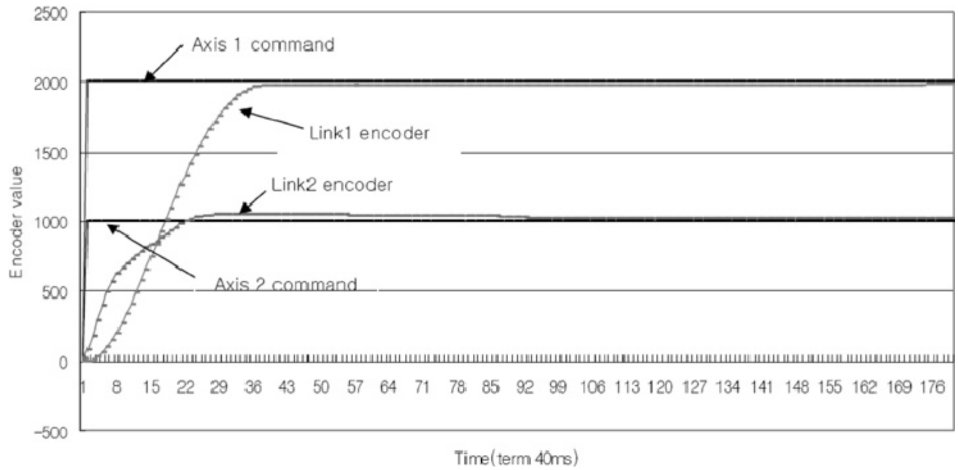

Fig. 3 Step response by PID control for a rigid joint

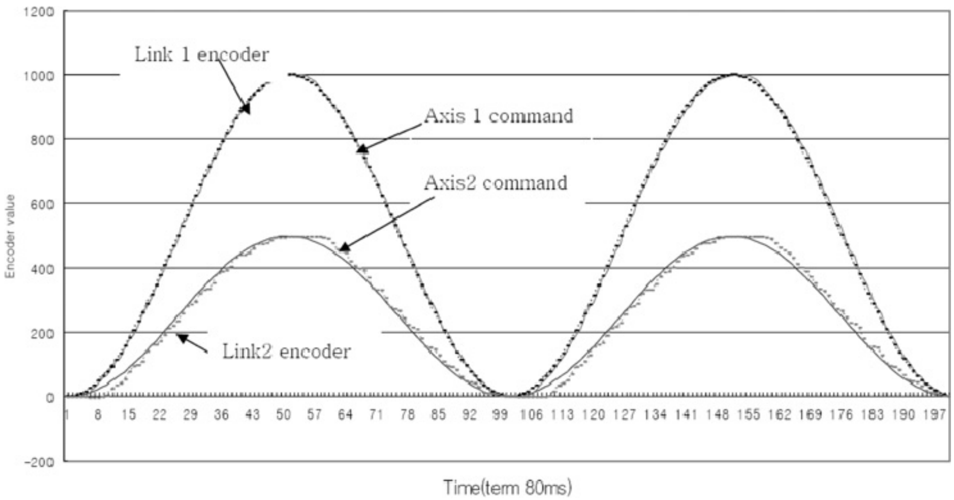

Fig. 4 Sinusoidal response with PID control for a rigid joint

puts. The PID gains for each link are assigned as P gain=100, D gain=10, I gain=20 for the link 1, and P gain=50, D gain=10, I gain=20 for the link 2. The gains are determined by several trials as long as the system performance is as desirable as possible. On the other hand, under different sets of the PID gains, the system shows undesirable behaviors such as high overshoot, instability, etc. which is mainly from the joint flexibility. The vertical axis for each figure shows number of the encoder pulses for the joint angle, and the encoder has 10,000 pulses for one rotation. For hard stiffness between the motor and link, which is recognized as an almost rigid manipulator, the control performance is desirable under PID or robust adaptive control viewing the experimental results. In general, for a rigid manipulator, it is known that the tracking performance is pretty acceptable by PID Control itself. However, when the stiffness between motor and link is small, whether the reference input is given by step or sinusoidal form, the system performance is not satisfactory due to a large steady state error and oscillation, which implies the appropriate motor torque is not enough delivered to the link due to a large decoupling between those (Figs. 5 and 7).It implies that the PID control does not compensate the uncertainty portions and structural decoupling. To justify the control performance by the adaptive robust control, a robust control scheme is

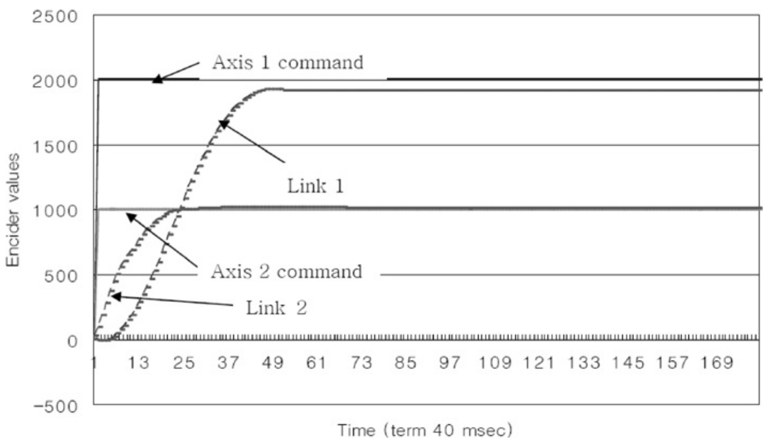

Fig. 5 Step response with PID control for flexible joint manipulator

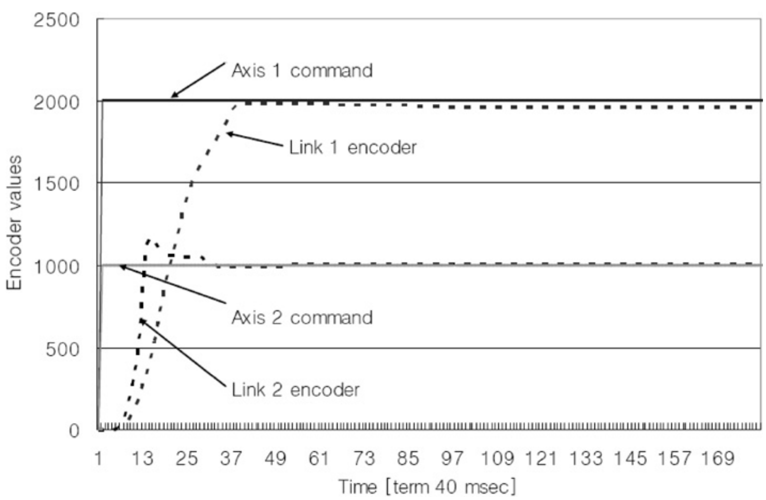

Fig. 6 Step response under adaptive robust control for flexible joint

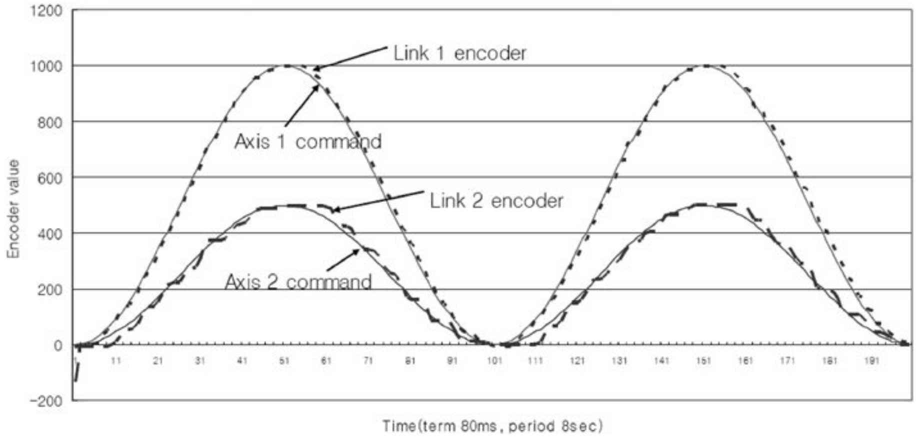

Fig. 7 Sinusoidal response with PID control for flexible joint

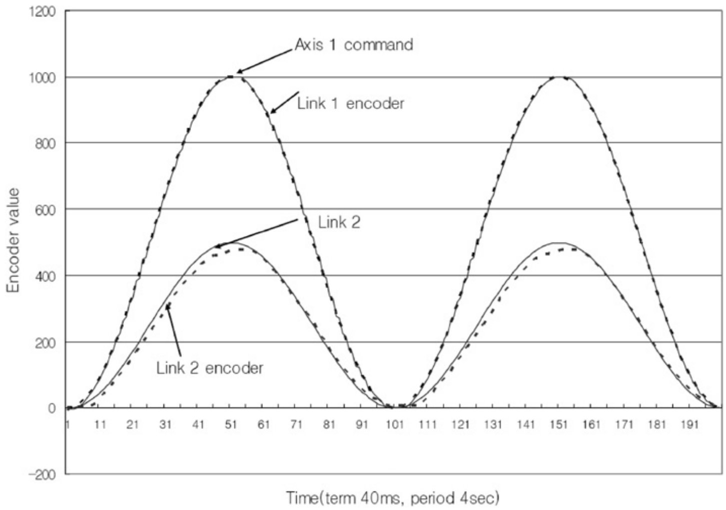

Fig. 8 Sinusoidal response with adaptive robust control for flexible joint

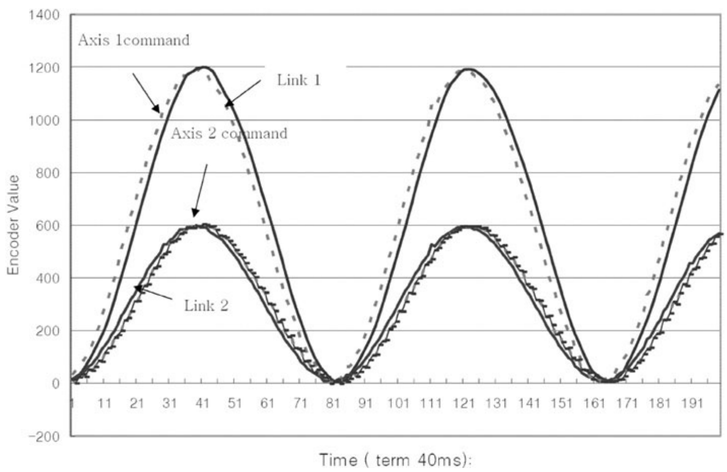

Fig. 9 Sinusoidal response with robust control for flexible joint

employed in Fig. 9. The steady state error is bigger than the error by adaptive robust control even if the steady state error could be decreased by selection of the control gains, which is similar to ε_1, and ε_2 in our controller set. However, severely small selection of the gains gives rise to a chattering in a control (Kim and Oho, 2006). In general, for a rigid manipulator, it is known that the tracking performance is somehow acceptable by PID control. However, when the stiffness between motor and link is small and soft, whether the reference input is given by a step or a sinusoidal form, the system performance is not satisfactory due to a large steady state error and fluctuation, which implies the motor motion does not much deliver to the link due to a large coupling between those. It implies that the PID control does not completely compensate the uncertainty portions and structural coupling.

On the other hand, with the use of the adaptive robust control, an improved system performance with respect to smaller settling time and steady state error is achieved in comparing with the PID control or robust control. Even under soft flexibility which is enough to cause a decoupling between the motor and the link, the control performance is very affirmative by adapting the adaptive robust control.

5. Conclusions

An adaptive robust control has been constructed for flexible joint manipulators which are nonlinear, time-varying and mismatched. State transformation via implanted control is introduced. No statistical property of the uncertainty is assumed and utilized. Only the existence of the bound of uncertainty is assured, although the bound is not given a priori. The scheme utilizes the bounding function by combining states and parameters, which is to be estimated. The controlled system has a nice property such that states for the transformed system converge to zero. Furthermore, the system guarantees uniform boundness, which is a stricter condition than a general adaptive control scheme. This is also true for the original system in case when either the gravitational force is absent or the system is coordinated such that the gravitational force converges to zero as link angles approach zero. However, since ε_1 and ε_2 will decay, we need to be careful when selecting l_1 and l_2 to overcome chattering in practical implementations. Also, in view of experimental work, sampling time of the control system needs to be carefully selected not to cause a chattering or unstability. The fact that the adaptive robust control is free of selecting the bounding function gains an advantage over robust control in real applications.

Appendix

Proof of Theorem 1. Let $\tilde{\beta}_1 := \hat{\beta}_1 - \beta_1$ and $\tilde{\beta}_2 := \hat{\beta} - \beta_2$ Choose functions $V_{1T}(z_2, \hat{\beta}_1)$ and $V_{2T}(z_2, \hat{\beta}_2)$ as follows:

$$V_{1T}(z_1, \tilde{\beta}_1) = V_1(z_1) + V_{\beta 1}(\tilde{\beta}_1) + l_1 \varepsilon_1, \tag{65}$$

$$V_{2T}(z_2, \tilde{\beta}_2) = V_2(z_2) + V_{\beta 2}(\tilde{\beta}_2) + l_2 \varepsilon_2, \tag{66}$$

Where

$$\begin{aligned} V_1(z_1) &= \frac{1}{2}(Z_2 + S_1 Z_1)^T \hat{D} (Z_2 + S_1 Z_1) \\ &\quad + \frac{1}{2} Z_1^T (K_{p1} + S_1 K_{v1}) Z_1, \\ V_{\beta 1}(\tilde{\beta}_1) &= \frac{1}{2}(\hat{\beta}_1 - \beta_1)^T T_1 (\hat{\beta}_1 - \beta_1), \\ V_2(z_2) &= \frac{1}{2}(Z_4 + S_2 Z_3)^T J (Z_4 + S_2 Z_3) \\ &\quad + \frac{1}{2} Z_3^T (K_{p2} + S_2 K_{v2}) Z_3, \\ V_{\beta 2}(\tilde{\beta}_2) &= \frac{1}{2}(\hat{\beta}_2 - \beta_2)^T T_3 (\hat{\beta}_2 - \beta_2). \end{aligned} \tag{67}$$

To show that V_{1T} and V_{2T} are legitimate Lyapunov function candidates, we prove that both V_{1T} and V_{2T} are positive definite and decrescent. Based on Assumption 3,

$$\begin{aligned} V_1(z_1) &\geq \frac{1}{2} \underline{\sigma}_k \| Z_2 + S_1 Z_1 \|^2 + \frac{1}{2} Z_1^T (K_{p1} + S_1 K_{v1}) Z_1 \\ &= \frac{1}{2} \underline{\sigma}_k \Sigma \sum_{i=1}^{n} (Z_{2i}^2 + 2 S_{1i} Z_{2i} Z_{1i} + S_{1i}^2 Z_{1i}^2) \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} (k_{p1i} + s_{1i} k_{v1i}) Z_{1i}^2 \\ &= \frac{1}{2} \sum_{i=1}^{n} [Z_{1i} \ z_{2i}] \underline{\Omega}_{1i} \begin{bmatrix} Z_{1i} \\ Z_{2i} \end{bmatrix}, \end{aligned} \tag{68}$$

$$V_{\beta_1}(\tilde{\beta}_1) \geq \frac{1}{2}\lambda_{\min}(T_1)\|\tilde{\beta}_1\|^2, \tag{69}$$

where

$$\underline{\Omega}_{1i} = \begin{bmatrix} \underline{\sigma}_k S_{1i}^2 + k_{p1i} + S_{1i}k_{v1i} & \underline{\sigma}_k S_{1i} \\ \underline{\sigma}_k S_{1i} & \underline{\sigma}_k \end{bmatrix}. \tag{70}$$

Here, Z_{1i} and Z_{2i} are the i-th components of Z_1 and Z_2, respectively. Since $\underline{\Omega}_{1i}>0$, $\forall i$ and $T_1>0$, V_1 and V_{β_i} are positive definite.

Next, in conjunction with Assumption 4 it can be seen that

$$\begin{aligned} V_1(z_1) &\leq \frac{1}{2}\bar{\sigma}_k\|Z_2+S_1Z_1\|^2 + \frac{1}{2}Z_1^T(K_{p1}+S_1K_{v1})Z_1 \\ &= \frac{1}{2}\bar{\sigma}_k\sum_{i=1}^{n}(Z_{2i}^2+2S_{1i}Z_{2i}Z_{1i}+S_{1i}^2Z_{1i}^2) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}(k_{p1i}+S_{1i}k_{v1i})Z_{1i}^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}[Z_{1i}Z_{2i}]\bar{\Omega}_{1i}\begin{bmatrix} Z_{1i} \\ Z_{2i}\end{bmatrix}, \end{aligned} \tag{71}$$

$$V_{\beta_1}(\tilde{\beta}_1) \leq \lambda_{\max}(T_1)\|\tilde{\beta}_1\|^2, \tag{72}$$

where

$$\bar{\Omega}_{1i} = \begin{bmatrix} \bar{\sigma}_k S_{1i}^2 + k_{p1i} + S_{1i}k_{v1i} & \bar{\sigma}_k S_{1i} \\ \bar{\sigma}_k S_{1i} & \bar{\sigma}_k \end{bmatrix}. \tag{73}$$

Consider the Lyapunov function candidate V_{1T}:

$$\begin{aligned} V_1 &\geq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\text{mix}}(\underline{\Omega}_{ii})(Z_{1i}^2+\dot{Z}_{1i}^2) \\ &\geq \gamma_1^{(1)}\|z_1\|^2, \end{aligned} \tag{74}$$

$$\begin{aligned} V_{\beta_1} &\geq \frac{1}{2}\lambda_{\min}(T_1)\|\hat{\beta}_1-\beta_1\|^2 \\ &= \gamma_7^{(1)}\|\hat{\beta}_1-\beta_1\|^2, \end{aligned} \tag{75}$$

where

$$\begin{aligned} \gamma_1^{(1)} &= \frac{1}{2}\min_i\{\min_{\sigma 1}\lambda_{\min}(\underline{\Omega}_{1i}),\ i=1,2,\cdots,n\}, \\ \gamma_7^{(1)} &= \frac{1}{2}\lambda_{\min}(T_1), \end{aligned} \tag{76}$$

and $\gamma_1^{(1)}, \gamma_7^{(1)}$ are unknown constants. Furthermore, we have

$$\begin{aligned} V_1 &\leq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\text{mix}}(\bar{\Omega}_{1i})(Z_{1i}^2+\dot{Z}_{1i}^2) \\ &\geq \gamma_2^{(1)}\|z_1\|^2, \end{aligned} \tag{77}$$

$$\begin{aligned} V_{\beta_1} &\leq \frac{1}{2}\lambda_{\max}(T_1)\|\hat{\beta}_1-\beta_1\|^2 \\ &= \gamma_8^{(1)}\|\hat{\beta}_1-\beta_1\|^2 \end{aligned} \tag{78}$$

where

$$\begin{aligned} \gamma_2^{(1)} &= \frac{1}{2}\max_i\{\max_{\sigma 1}\lambda_{\max}(\bar{\Omega}_{1i}),\ i=1,2,\cdots,n\}, \\ \gamma_8^{(1)} &= \frac{1}{2}\lambda_{\max}(T_1), \end{aligned} \tag{79}$$

and $\gamma_2^{(1)}, \gamma_8^{(1)}$ are unknown constants.

Similar to V_1, V_2 can be shown to satisfy

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{n}\lambda_{\min}(\underline{\Omega}_{2i})(Z_{3i}^2+\dot{Z}_{3i}^2) \leq V_2(z_2) \\ &\leq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\max}(\underline{\Omega}_{2i})(Z_{3i}^2+\dot{Z}_{3i}^2), \end{aligned} \tag{80}$$

$$\begin{aligned} &\frac{1}{2}\lambda_{\min}(T_2)\|\hat{\beta}_2-\beta_2\|^2 \leq V_{\beta_2}\left(\hat{\beta}_2\right) \\ &\leq \frac{1}{2}\lambda_{\max}(T_2)\|\hat{\beta}_2-\beta_2\|^2, \end{aligned} \tag{81}$$

Thus

$$\gamma_2^{(2)}\|z_2\|^2 \leq V_2(z_2) \leq \gamma_2^{(2)}\|z_2\|^2, \tag{82}$$

$$\lambda_7^{(2)}\|\hat{\beta}_2-\beta_2\|^2 \leq V_{\beta_2}(\hat{\beta}_2) \leq \gamma_8^{(2)}\|\hat{\beta}_2-\beta_2\|^2, \tag{83}$$

where

$$\begin{aligned} \gamma_1^{(2)} &= \frac{1}{2}\min_i\{\min_{\sigma 1}\lambda_{\min}(\bar{\Omega}_{2i}),\ i=1,2,\cdots,n\}, \\ \gamma_2^{(2)} &= \frac{1}{2}\max_i\{\max_{\sigma 1}\lambda_{\max}(\bar{\Omega}_{2i}),\ i=1,2,\cdots,n\}, \\ \gamma_7^{(2)} &= \frac{1}{2}\lambda_{\min}(T_2), \\ \gamma_8^{(2)} &= \frac{1}{2}\lambda_{\max}(T_2), \end{aligned} \tag{84}$$

also,

$$\begin{aligned} \underline{\Omega}_{2i} &:= \begin{bmatrix} \underline{\theta}S_{2i}^2 + k_{p2i} + S_{2i}k_{v2i} & \underline{\theta}S_{21} \\ \underline{\theta}S_{2i} & \underline{\theta} \end{bmatrix}, \\ \bar{\Omega}_{2i} &:= \begin{bmatrix} \bar{\theta}S_{2i}^2 + k_{p2i} + S_{2i}k_{v2i} & \bar{\theta}S_{21} \\ \bar{\theta}S_{2i} & \bar{\theta} \end{bmatrix}, \\ \bar{\theta} &:= \lambda_{\max}(J), \\ \underline{\theta} &:= \lambda_{\min}(J), \end{aligned} \tag{85}$$

and $\gamma_1^{(2)}, \gamma_2^{(2)}$ are unknown constants. The derivative of V_{1T} is given by

$$\dot{V}_{1T} = \dot{V} + \dot{V}_{\beta 1} + l_1\dot{\varepsilon}_1. \tag{86}$$

Concerning $\dot{V}_1$, it can be seen that

$$\begin{aligned} \dot{V}_1 &= (\dot{Z}_1+S_1Z_1)^T\hat{D}(\ddot{Z}_1+S_1Z_1)^T \\ &\quad + \frac{1}{2}(\dot{Z}_1+S_1Z_1)^T\dot{\hat{D}}(\dot{Z}_1+S_1Z_1) \\ &\quad + Z_1^T(K_{p1}+S_1K_{v1})\dot{Z}_1. \end{aligned} \tag{87}$$

From (42), we obtain

$$\begin{aligned}\dot{V}_1 &= (\dot{Z}_1+S_1Z_1)^T(-\dot{C}\dot{Z}_1-\hat{G}-Z_1+Z_3+u_1\\ &\quad +\hat{D}S_1Z_1+\frac{1}{2}\dot{\hat{D}}\dot{Z}_1+\frac{1}{2}\dot{\hat{D}}S_1Z_1)\\ &\quad +Z_1^T(K_{p1}+S_1K_{v1})\dot{Z}_1\\ &=(\dot{Z}_1+S_1Z_1)^T(\frac{1}{2}\dot{\hat{D}}(\dot{Z}_1+S_1Z_1)-\dot{C}\dot{Z}_1\\ &\quad -\hat{G}-Z_1+\hat{D}S_1\dot{Z}_1)\\ &\quad +(\dot{Z}_1+S_1Z_1)^Tu_1+(\dot{Z}_1+S_1Z_1)^TZ_3\\ &\quad +Z_1^T(K_{p1}+S_1K_{v1})\dot{Z}_1.\end{aligned} \tag{88}$$

Using the "control" u_1 in (17), (13), and (15) it can be shown that

$$\begin{aligned}\dot{V}_1 &\leq (\dot{Z}_1+S_1Z_1)^T(-K_{p1}Z_1-K_{v1}\dot{Z}_1+p_1)\\ &\quad +\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)+(\dot{Z}_1+S_1Z_1)^TZ^3\\ &\quad +Z_1^T(K_{p1}+S_1K_{v1})\dot{Z}_1\\ &\leq -\underline{\lambda}\|Z_1\|^2+\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)\\ &\quad +(\dot{Z}_1+S_1Z_1)^Tp_1+\|\dot{Z}_1+S_1Z_1\|\|Z_3\|.\end{aligned} \tag{89}$$

For $\|\mu_{1i}\|>\varepsilon_1$, the second and third term in (89) becomes

$$\begin{aligned}&\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)+(\dot{Z}_1+S_1Z_1)^Tp_1\\ &\sum_{i=1}^{n}\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|\Pi_i(z_1,\beta_1)\\ &+\sum_{i=1}^{n}(\dot{Z}_{1i}+S_{1i}Z_{1i})\Big(-\frac{\dot{Z}_{1i}+S_{1i}Z_{1i}}{\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|}\Pi_1(z_1,\hat{\beta}_1)\Big)\\ &=\sum_{i=1}^{n}\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|(\Pi_1(z_1,\hat{\beta}_1)).\end{aligned} \tag{90}$$

When $\|\mu_{1i}\|\leq\varepsilon_1$, then

$$\begin{aligned}&\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)+(\dot{Z}_1+S_1Z_1)^Tp_1\\ &\leq\sum_{i=1}^{n}\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|\Pi_i(z_1,\beta_1)\\ &\quad+\sum_{i=1}^{n}(\dot{Z}_{1i}+S_{1i}Z_{1i})^2\Big(-\Pi_1^2(z_1,\hat{\beta}_1)\frac{1}{\varepsilon_1}\Big).\end{aligned} \tag{91}$$

Concerning $\dot{V}_{\beta_1}$, it follows from (21)

$$\begin{aligned}\dot{V}_{\beta_1}&=(\hat{\beta}_1-\beta_1)^TT_1\dot{\hat{\beta}}_1\\ &=(\hat{\beta}_1-\beta_1)^T\frac{\partial\Pi_1^T}{\partial\beta_1}(z_1,\hat{\beta}_1)\|\dot{Z}_1+S_1Z_1\|\end{aligned} \tag{92}$$

Since $-\Pi_1(z_1,\cdot)$ is convex for all $Z_2\in R^{2n}$, it is true that

$$\frac{\partial\Pi_1}{\partial\beta_1}(z_1,\hat{\beta}_1)(\hat{\beta}_1)\leq\Pi_1(z_1,\hat{\beta}_1)-\Pi_1(z_1,\beta_1), \tag{93}$$

therefore, we get

$$\dot{V}_{\beta 1}\leq(\Pi_1(z_1,\hat{\beta}_1)-\Pi_1(z_1,\beta_1))\|\dot{Z}_1+S_2Z_1\|. \tag{94}$$

Finally, by (90) and (91) $\dot{V}_1$ is upper-bounded by :

$$\begin{aligned}\dot{V}_{1T}&\leq-\underline{\lambda}_1\|z_1\|^2+\sum_{i=1}^{n}\|\dot{Z}_{1i}+S_{1i}z_{1i}\|\Pi_1(z_1,\hat{\beta}_1)\\ &\quad-\sum_{i=1}^{n}(\dot{Z}_{1i}+S_{1i}Z_{1i})^2\Pi_1^2(z_1,\hat{\beta}_1)\frac{1}{\varepsilon_1}+l_1\dot{\varepsilon}_1\\ &\quad+\|\dot{Z}_1+S_1Z_1\|\|Z_3\|\\ &\leq-\underline{\lambda}_1\|z_1\|^2+\frac{n\varepsilon_1}{4}-\frac{n\varepsilon_1}{4}+\|\dot{Z}_1+S_1Z_1\|\|Z_3\|\\ &\leq-\underline{\lambda}_1\|z_1\|^2+\frac{1}{2}\omega_1\lambda_{\max}(\bar{S}_1)\|z_1\|^2+\frac{1}{2}\omega_1^{-1}\|z_2\|^2\\ &\leq-\Big(\underline{\lambda}_1-\frac{1}{2}\omega_1\lambda_{\max}(\bar{S}_1)\Big)\|z_1\|^2+\frac{1}{2}\omega_1^{-1}\|z_2\|^2.\end{aligned} \tag{95}$$

Next, the derivative of V_{2T} is given by

$$\dot{V}_{2T}=\dot{V}_2+\dot{V}_{\beta 2}+l_2\dot{\varepsilon}_2. \tag{96}$$

Concerning $\dot{V}_2$, it follows from (12) and (26)

$$\begin{aligned}\dot{V}_2&=(\dot{Z}_3+S_2Z_3)^TJ(\ddot{Z}_3+S_2\dot{Z}_3)+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3\\ &=(\dot{Z}_3+S_2Z_3)^T(-J\ddot{u}_1-KZ_3+KZ_1-Ku_1+JS_2\dot{Z}_3+u)\\ &\quad+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3\\ &=(\dot{Z}_3+S_2Z_3)^T(\Phi_2+u)+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3.\end{aligned} \tag{97}$$

It follows from (27-29)

$$\begin{aligned}\dot{V}_2&\leq\|\dot{Z}_3+S_2Z_3\|\|\phi_2\|+(\dot{Z}_3+S_3Z_3)^Tu\\ &\quad+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3\\ &\leq\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\beta_2)\\ &\quad+(\dot{Z}_3+S_2Z_3)^T(-K_{p2}Z_3-K_{v2}\dot{Z}_3+p_2)\\ &\quad+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3\\ &\leq-\underline{\lambda}_2\|z_2\|^2+\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\beta_2)\\ &\quad+(\dot{Z}_3+S_2Z_3)^Tp_2.\end{aligned} \tag{98}$$

Concerning $\dot{V}_{\beta_2}$, it follows from (32)

$$\begin{aligned}\dot{V}_{\beta_2}&=\tilde{\beta}_2^TT_2\dot{\hat{\beta}}_2\\ &=(\beta_1-\beta_1)^TT_2\Big(T_2^{-1}\frac{\partial\Pi_2^T}{\partial\beta_2}(z_1,z_2,\hat{\beta}_2)\Big)\|\dot{Z}_3+S_2Z_3\|.\end{aligned} \tag{99}$$

Owing to the condition that $-\Pi_2(z_1,z_2,\cdot)$ is convex for all $z_1,z_2\in R^{2n}$,

$$\dot{V}_{\beta_2}\leq(\Pi_2(z_1,z_2,\hat{\beta}_2)-\Pi_2(z_1,z_2,\beta_2))\|\dot{Z}_3+S_2Z_3\|. \tag{100}$$

By adding (100), we obtain

$$\begin{aligned}\dot{V}_{2T}&\leq(\dot{Z}_3+S_2Z_3)^T\Pi_2(z_1,z_2,\hat{\beta}_2)+(\dot{Z}_3+S_2Z_3)^Tu\\ &\quad+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3+l_2\dot{\varepsilon}_2.\end{aligned} \tag{101}$$

From the control u in (29), it can be seen that

$$\begin{aligned}\dot{V}_{2T}&=(\dot{Z}_3+S_2Z_3)^T\Pi_2(z_1,z_2,\hat{\beta}_2)\\ &\quad+(\dot{Z}_3+S_2Z_3)^T(-K_{p2}Z_3-K_{v2}\dot{Z}_3+p_2)\\ &\quad+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3+l_2\dot{\varepsilon}_2\\ &\leq-\underline{\lambda}_2\|z_2\|^2+\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\hat{\beta}_2)\\ &\quad+(\dot{Z}_3+S_2Z_3)^Tp_2+l_2\dot{\varepsilon}_2.\end{aligned} \tag{102}$$

For $\| \mu_2(z_1, z_2, \hat{\beta}_2) \| > \varepsilon_2$, the second and third term in (102) becomes

$$\begin{aligned} &\| \dot{Z}_3 + S_2 Z_3 \| \Pi_2(z_1, z_2, \beta_2) + (\dot{Z}_3 + S_2 Z_3)^T p_2 \\ &= \| \dot{Z}_3 + S_2 Z_3 \| \Pi_2(z_1, z_2, \beta_2) \\ &\quad - \| \dot{Z}_3 + S_2 Z_3 \| \Pi_2(z_1, z_2, \hat{\beta}_2) \\ &= 0. \end{aligned} \tag{103}$$

When $\| \mu_2(z_1, z_2, \hat{\beta}_2) \| \le \varepsilon_2$, then

$$\begin{aligned} &\| \dot{Z}_3 + S_2 Z_3 \| \Pi_2(z_1, z_2, \beta_2) + (\dot{Z}_3 + S_2 Z_3)^T p_2 \\ &= \| \dot{Z}_3 + S_2 Z_3 \| \Pi_2(z_1, z_2, \beta_2) \\ &\quad - \| \dot{Z}_3 + S_2 Z_3 \|^2 \Pi_2^2(z_1, z_2, \hat{\beta}_2) \frac{1}{\varepsilon_2} \le \frac{\varepsilon_2}{4}. \end{aligned} \tag{104}$$

Therefore, we have

$$\begin{aligned} \dot{V}_{2T} &\le -\underline{\lambda}_2 \| z_2 \|^2 + \frac{\varepsilon_2}{4} + l_2 \dot{\varepsilon}_2 \\ &= -\underline{\lambda}_2 \| z_2 \|^2 + \frac{\varepsilon_2}{4} + l_2 \left(-\frac{\varepsilon_2}{4 l_2} \right) \\ &= \underline{\lambda}_2 \| z_2 \|^2 \end{aligned} \tag{105}$$

This shows that $\dot{V}_{2T}$ is bounded from above. By above results (95) and (105), we get

$$\begin{aligned} \dot{V}_T &= \dot{V}_1{}^T + \dot{V}_{2T} \\ &\le -\left(\underline{\lambda}_1 - \frac{1}{2} \omega_1 \lambda_{\max}(\bar{S}_1) \right) \| z_1 \|^2 \\ &\quad - \left(\underline{\lambda}_2 - \frac{1}{2} \omega_1^{-1} \right) \| z_2 \|^2, \end{aligned} \tag{106}$$

If we choose $\underline{\lambda}_1$ and $\underline{\lambda}_2$ such that

$$\begin{aligned} &\underline{\lambda}_1 - \frac{1}{2} \omega_1 \lambda_{\max}(\bar{S}_1) > 0, \\ &\underline{\lambda}_1 - \frac{1}{2} \omega_1^{-1} > 0, \end{aligned} \tag{107}$$

then we have

$$\begin{aligned} &\dot{V}_T \le -\min\left\{ \underline{\lambda}_1 - \frac{1}{2} \omega_1 \lambda_{\max}(\bar{S}_1),\ \underline{\lambda}_2 - \frac{1}{2} \omega_1^{-1} \right\} \| z_2 \|^2 \\ &=: -\gamma_3(\| z \|)\ a.e. on (t_0, t_1), \end{aligned} \tag{108}$$

where

$$r_3(\| z \|) = \min\left\{ \underline{\lambda}_1 - \frac{1}{2} \omega_{\max}(\bar{S}_1),\ \underline{\lambda}_2 - \frac{1}{2} \omega_1^{-1} \right\} \| z_2 \|^2. \tag{109}$$

This concludes boundness of the states and the states convergence to zero is now investigated. Consider any $\eta > 0$ let

$$\begin{aligned} &\underline{\gamma}_1(\| z \|) = \min\{ \gamma_1^{(1)}, \gamma_1^{(2)} \} \| z \|^2, \\ &\underline{\gamma}_2(\| z \|) = \max\{ \gamma_2^{(1)}, \gamma_2^{(2)} \} \| z \|^2, \\ &\gamma_7(\| \hat{\beta} - \beta \|) = \min\{ \gamma_7^{(1)}, \gamma_7^{(2)} \} \| \hat{\beta} - \beta \|^2, \\ &\gamma_8(\| \hat{\beta} - \beta \|) = \max\{ \gamma_8^{(1)}, \gamma_8^{(2)} \} \| \hat{\beta} - \beta \|^2 \end{aligned}$$

where

$$\hat{\beta} = [\hat{\beta}_1^T\ \hat{\beta}_2^T]^T,\ \beta = [\beta_1^T\ \beta_2^T]^T$$

and

$$\bar{\eta} = \min\{ \underline{\gamma}_1(\eta),\ \gamma_7(\eta) \}.$$

We are choose $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} &\| z(t_0) \| < \delta_1 \Rightarrow \underline{\gamma}_2(\| z(t_0) \|) < \frac{\bar{\eta}}{2}, \\ &\| \hat{\psi}(t_0) - \psi \| < \delta_2 \Rightarrow \gamma_8(\| \hat{\psi}(t_0) - \psi \|) < \frac{\bar{\eta}}{2} \end{aligned} \tag{110}$$

and take $\delta = \min\{ \delta_1, \delta_2 \}$. Now, we define

$$\bar{\gamma}_1^{-1} := \{ r \in R_+ :\ \underline{\gamma}_i(r) = s \}\ \forall s \in [0, \infty],\quad i = 1, 2, \tag{111}$$

then, we get

$$\begin{aligned} &\bar{\gamma}_1^{-1}(s) = \left(\frac{2s}{\min\{ \gamma_1^{(1)}, \gamma_1^{(2)} \}} \right)^{\frac{1}{2}}, \\ &\bar{\gamma}_2^{-1}(s) = \left(\frac{2s}{\min\{ \gamma_2^{(1)}, \gamma_2^{(2)} \}} \right)^{\frac{1}{2}}. \end{aligned} \tag{112}$$

For any r_1, r_2 let

$$\begin{aligned} &d_1(r_1, r_2) = \bar{\gamma}_1^{-1}[\underline{\gamma}_2(r_1) + \underline{\gamma}_8(r_2)], \\ &d_2(r_1, r_2) = \bar{\gamma}_2^{-1}[\underline{\gamma}_2(r_1) + \underline{\gamma}_8(r_2)], \end{aligned} \tag{113}$$

With η and $d_1(r_1, r_2)$, $d_2(r_1, r_2)$ given above, Properties 2-4 follow directly as we apply the results (Corless and Leitmann, 1984). For the Property 5, we consider from the results of $\dot{V}_1$ and $\dot{V}_2$

$$\begin{aligned} \dot{V}_1 + \dot{V}_2 &\le \| \dot{Z}_1 + S_1 Z_1 \| \Pi_1(z_1, \beta_1) + (\dot{Z}_1 + S_1 Z_1)^T p_1 - \lambda \| z_1 \|^2 \\ &\quad + \frac{1}{2} \lambda_{\min}(\bar{S}_1) w_1 \| z_1 \|^2 + \frac{1}{2} w_1^{-1} \| z_2 \|^2 - \underline{\lambda}_2 \| z_2 \|^2 \\ &\quad + \| \dot{Z}_3 + S_2 Z_3 \| \Pi_2(z_1, z_2, \beta_2) + (\dot{Z}_3 + S_2 Z_3)^T p_2 \\ &\le \| \dot{Z}_1 + S_1 Z_1 \| \Pi_1(z_1, \beta_1) + (\dot{Z}_1 + S_1 Z_1)^T p_1 \\ &\quad + \| \dot{Z}_3 + S_2 Z_3 \| \Pi_2(z_1, z_2, \beta_2) + (\dot{Z}_3 + S_2 Z_3)^T p_2 \end{aligned} \tag{114}$$

For the first and second term of (114) it can be seen that

$$\begin{aligned} &\| \dot{Z}_1 + S_1 Z_1 \| \Pi_1(z_1, \beta_1) + (\dot{Z}_1 + S_1 Z_1)^T p_1 \\ &= \| \dot{Z}_1 + S_1 Z_1 \| (\Pi_1(z_1, \beta_1) - \Pi_1(z_1, \hat{\beta}_1)) \\ &\quad + \| \dot{Z}_1 + S_1 Z_1 \| \Pi_1(z_1, \hat{\beta}_1) + (\dot{Z}_1 + S_1 Z_1)^T p_1 \end{aligned} \tag{115}$$

For the first two terms in (114) we get

$$\begin{aligned} &\| \dot{Z}_1 + S_1 Z_1 \| (\Pi_1(z_1, \beta_1) - \Pi_1(z_1, \hat{\beta}_1)) \\ &\le \| \dot{Z}_1 + S_1 Z_1 \| \frac{\partial \Pi_1}{\partial \beta_1}(z_1, \hat{\beta}_1)(\beta_1 - \hat{\beta}_1) \\ &\le \sum_{i=1}^{n} \| \dot{Z}_1 + S_1 Z_1 \| \pi(\beta_{1i} - \hat{\beta}_{1i}) \frac{\partial \Pi_1}{\partial \beta_1}(z_1, \hat{\beta}_1) \\ &=: a_1(t), \end{aligned} \tag{116}$$

Where $\pi : R \rightarrow R_+$ is given by

$$\pi(a) = \begin{cases} 0, & a<0 \\ a, & a>0 \end{cases} \quad (117)$$

We see that $a_1(t) \geq 0$ for all $t \in [t_0, \infty)$ Also utilizing (116) and (21), we obtain for each $t \in [t_0, \infty)$.

$$\int_0^t a_1(\tau)\,d\tau = \int_0^t \sum_{i=1}^n \pi(\beta_{1i} - \hat{\beta}_{1i})\, T_1 \hat{\beta}_{1i}(\tau)\,d\tau = \eta(t_0) - \eta(t), \quad (118)$$

where

$$\eta(t) := \sum_{i=1}^n \frac{1}{2} T_1 \pi^2(\beta_{1i} - \hat{\beta}_{1i}(t)). \quad (119)$$

Since $\eta(t) \geq 0$, we see that

$$\int_0^t a_1(\tau)\,d\tau \leq \eta(t_0)\ \forall t \in [t_0, \infty) \quad (120)$$

Hence, $\int_0^\infty a_2(\tau)\,d\tau$ is finite. For the last two terms of (115) it can be shown

$$\|\dot{Z}_1 + S_1 Z_1\| \Pi_1(z_1, \beta_1) + (\dot{Z}_1 + S_1 Z_1)^T p_1 \leq \frac{n\varepsilon_1(t)}{4} \leq \frac{n\varepsilon_1(t_0)}{4} \quad (121)$$

Therefore, by (116) and (121) the first two terms in (128) are given by

$$\|\dot{Z}_1 + S_1 Z_1\| \Pi_1(z_1, \beta_1) + (\dot{Z}_1 + S_1 Z_1)^T p_1 \leq a_1(t) \frac{n\varepsilon_1(t_0)}{4}. \quad (122)$$

Similar to (122), the following can be seen to satisfy

$$\|\dot{Z}_3 + S_2 Z_3\| \Pi_2(z_1, z_2, \beta_2) + (\dot{Z}_3 + S_2 Z_3)^T p_2 \leq a_2(t) + \frac{\varepsilon_2(t_0)}{4}, \quad (123)$$

where

$$a_2(t) := \sum_{i=1}^n \pi(\beta_{1i} - \hat{\beta}_{1i}) \frac{\partial \Pi_2}{\partial \beta_{2i}} (z_1, z_2, \hat{\beta}_2) \|\dot{Z}_3 + S_2 Z_3\|. \quad (124)$$

Also, we see that $a_2(t) \geq 0$ and $\int_0^\infty a_2(\tau)\,d\tau$ is finite. Therefore, we get

$$\dot{V}_1 + \dot{V}_2 \leq a_1(t) + \frac{n\varepsilon_1(t_0)}{4} + a_2(t) + \frac{\varepsilon_2(t_0)}{4} =: b_1(t) + b_2, \quad (125)$$

Where

$$b_1(t) = a_1(t) = a_1(t), \quad b_2 = \frac{n\varepsilon_1(t_0)}{4} + \frac{\varepsilon_2(t_0)}{4}. \quad (126)$$

From the bound of $V_1(\cdot)$ and $V_2(\cdot)$, we obtain

$$\hat{\gamma}_1 \|z\|^2 \leq V_1(z_1) + V_2(z_2) \leq \hat{\gamma}_2 \|z\|^2, \quad (127)$$

where $\hat{\gamma} = \min(\gamma_1^{(1)}, \gamma_1^{(2)})$, $\hat{\gamma}_2 = \max(\gamma_2^{(1)}, \gamma_2^{(2)})$ In view of (108), we have the following result

$$\int_0^t \gamma_3(\|z(\tau)\|)\,d\tau \leq V_T(t_0) - V_T(t) \leq V_T(t_0) \quad (128)$$

for all $t \in [t_0, \infty)$ Hence, $\int_0^t \gamma_3(\|z(\tau)\|)\,d\tau$ is finite. Here, we see that $\int_0^\infty b_1(\tau)\,d\tau$ is finite, and $b_2(t) \geq 0$, $\forall t \in [t_0, \infty)$. The results of (127), (128), and finiteness of $\int_0^\infty \gamma_3(\|z(\tau)\|)\,d\tau$ and $\int_0^t b_1(\tau)\,d\tau$ satisfy Lemma 3 (Corless and Leitmann, 1984). Hence $\lim_{t\to\infty} \|z(t)\| = 0$, Thus, $\lim_{t\to\infty} z(t) = 0$, and we see that this fact satisfies Property 5.

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